MULTIVARIATE IMMUNIZATION THEORY

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ABSTRACT

Extending the general nonparallel shift approach to duration analysis developed previously [28], this paper explores the immunization properties of that model. In particular, results are developed for directional immunization, in which the yield curve shift direction vector is specified, as well as for nondirectional immunization. Throughout, the goal of immunization at a time k periods into the future is seen to be intimately linked to the relationship between the durational and convexity attributes of the portfolio and those of a k-period zero-coupon bond. Applications to asset/liability management are then explored in theory and in a detailed example, which illustrates the potential shortcomings of traditional parallel shift immunization.

I. INTRODUCTION

The concepts of duration and immunization have been the subjects of increasing interest, from both a theoretical and an applied perspective. Originally discovered more than 50 years ago, duration was defined to better reflect the length of a payment stream (Macaulay [21]). A short time later (Hicks [14]), it was independently derived in an investigation into the elasticity of the price of a bond with respect to the discount factor $v=(1+i)^{-1}$.

Soon thereafter (Samuelson [30]), Redington [23]), duration was rediscovered in the context of the immunization of a firm's or portfolio's net worth, that is, in pursuit of conditions under which assets and liabilities would be equally responsive to changes in an underlying interest rate. Redington's approach [23] was later adapted by Vanderhoof [34] and became what to many of today's actuaries represented an introduction to this field of thought and its application to insurance company portfolios. Common to the above investigations was the assumption of a single interest rate for all discountings of cash flows, that is, a flat yield curve.

Fisher and Weil [11] first extended the Redington model to reflect a nonflat term structure and developed a corresponding duration measure sometimes denoted D_2, to distinguish it from the Macaulay duration, D_1. This measure reflected price sensitivity to parallel shifts in the term structure, that is, shifts for which each yield point moves by the same amount.

Other definitions of duration were developed (Bierwag [3], Khang [20], Brennan and Schwartz [8]) corresponding to other models of yield curve dynamics, or the manner in which the term structure changes. Surveys of these models and related matters can be found in Bierwag, Kaufman and Toevs [7] and in Bierwag [2], who also provides a broad survey of many aspects of this theory and its applications.

The fact that immunization against a given yield curve shift assumption generally fails to provide protection against more general yield curve shifts was noted by Ingersoll, Skelton and Weil [17], Fong and Vasicek [12], and Shiu [31]. The importance of the correct choice for yield curve dynamics was noted in Milgrom [22], as well as in Bierwag, Kaufman and Toevs [4], who investigated stochastic process risk and demonstrated that losses associated with choosing the wrong model can be substantial.

Other extensions of Redington's work include that of Grove [13], who immunized a non-zero initial net worth; Kaufman [19], who investigated the immunization of the net worth asset ratio; and Bierwag, Kaufman and Toevs [5,6], who introduced a methodology for developing an immunizing asset portfolio and investigated the concept of an efficient frontier in this context.

More recent approaches have involved immunizing multiple liabilities (Shiu [32]), tax-adjusting the duration measure (Stock and Simonson [33]), and utilizing a duration vector approach to immunization (Chambers, Carleton and McEnally [9]). This last approach defines a vector in which the components reflect "moments" of adjusted times-to-receipt of the underlying cash flows. In this context, traditional duration is closely related to their first moment, while the concepts of convexity and inertia (Bierwag [2]) are closely related to their second moment. The adjustment made to the times-to-receipt of the cash flows is a reduction by one time unit.

A general nonparallel shift approach to duration analysis was developed in Reitano [24,28], and applications to measuring potential yield curve risk were given in Reitano [25,27,29]. For this analysis, the yield curve is identified with a vector of values representing the "yield curve drivers," which can be taken, for instance, as the yields at the commonly quoted maturities. The underlying technique employed is a general multivariate analysis. Although multivariate models are not new (Bierwag [2], Ho [16]), the general model utilized provides great insight to portfolio sensitivity to general yield curve shifts.

In particular, "partial" durations are defined to reflect yield sensitivities point by point along the yield curve. These measures are then easily combined to produce "directional" duration measures that reflect portfolio sensitivity to any yield curve shift. The traditional duration measure, for example, reflecting sensitivity in the parallel shift direction, is seen to be the sum of the underlying partial durations.

The current article extends this theory to the question of immunization. The yield curve is again modeled as a vector of yields, with other yields assumed to be functionally dependent, such as via interpolation. Consequently, all yield curve changes are identified with vector shifts, and immunization is pursued within this multivariate context.

This immunization model is introduced in Section II, along with the necessary definitions from Reitano [24,28]. Section III then develops the theory of "directional" immunization at time k, which is seen to be a natural extension of Redington's parallel shift approach to general but specified nonparallel yield curve shifts.

In this context, as throughout the paper, the goal of immunization at a time k periods into the future is seen to be intimately connected to the relationship of the portfolio's directional duration and convexity attributes to those of a k-period zero-coupon bond. Naturally, immunization results for the special case of parallel shifts are seen to be equivalent to well-known results. Also in this section, the concept of an immunization boundary is explored, extending the idea of duration window (Bierwag [2]), as is the portfolio return on investment, generalizing Babcock [1].

Section IV applies these general results to the context of asset/liability management. Surplus immunization conditions are developed in both the absolute and asset ratio contexts and the results translated to implications for the immunization boundary. An example is then developed in detail that demonstrates that immunization for one direction, for example, against parallel shifts, may provide little protection against more general shifts. This result is shown in theory and by using actual yield curve shifts from August 1984 through June 1990.

Section V then develops immunization results in the general nondirectional context, that is, conditions under which portfolio values at time k are preserved under all yield curve shifts. General return on investment results are also developed, as are the implications for asset/liability management.

Section VI investigates the relationship of immunization properties to the yield curve model employed.

A technical appendix contains the proofs of the duration theory underlying the immunization results.

II. MULTIVARIATE IMMUNIZATION

A. Multivariate Price Model

Let $P(\mathbf{i})$ denote a positive valued multivariate price function that reflects the dependency of the price of a portfolio of securities on an underlying yield curve vector, $\mathbf{i}=(i_1, \ldots, i_m)$. This portfolio could equally well reflect assets, liabilities, or a net worth or surplus position. The cash flows encompassed by $P(\mathbf{i})$ may be fixed or interest-dependent, with $P(\mathbf{i})$ correspondingly representing a simple present value price function, or the price values obtained via a model that incorporates the options or other interest dependencies (for example, Clancy [10], Ho and Lee [15], and Jacob, Lord and Tilley [18]).

The yield curve above is modeled as a discrete vector, representing the yield curve drivers in a given valuation model, which can be taken as the yields at the commonly quoted maturity points. This yield curve may reflect any system of units (bond yields, spot or forward rates) and any nominal basis (annual, semiannual, and so on). In practice, yield points at other maturities are typically derived from these values via interpolation and/or other conversion, so it is appropriate to view the price of the portfolio, $P(\mathbf{i})$, as a function of this yield curve vector. For example, with $\mathbf{i}$ reflecting bond yields, pivotal yield values for maturities 0.25, 1, 2, 3, 4, 5, 7, 10, 20, and 30 years are sufficient for most valuations, and $P(\mathbf{i})$ can be modeled as a function of these ten observed values.

As in Reitano [24,28], we make the following definitions, which generalize the notions of duration and convexity to this yield vector basis. Accordingly, we assume throughout that $P(\mathbf{i})$ is twice differentiable, with continuous second-order partial derivatives.

Definition 1

Given $P(\mathbf{i})$, the j-th *partial duration function*, denoted $D_j(\mathbf{i})$, and the jk-th *partial convexity function*, denoted $C_{jk}(\mathbf{i})$, are defined for $P(\mathbf{i}) \neq 0$ as follows:

$$D_j(\mathbf{i}) = -d_j P(\mathbf{i})/P(\mathbf{i}), \quad j = 1, \ldots, m \tag{2.1}$$

$$C_{jk}(\mathbf{i}) = d_{jk} P(\mathbf{i})/P(\mathbf{i}), \quad j, k = 1, \ldots, m \tag{2.2}$$

where $d_j P(\mathbf{i})$ and $d_{jk} P(\mathbf{i})$ denote the corresponding partial derivatives of $P(\mathbf{i})$.

The *total duration vector*, denoted $\mathbf{D}(\mathbf{i})$, and *total convexity matrix*, denoted $\mathbf{C}(\mathbf{i})$, are defined as follows:

$$\mathbf{D}(\mathbf{i}) = (D_1(\mathbf{i}), \ldots, D_m(\mathbf{i})), \tag{2.3}$$

$$\mathbf{C}(\mathbf{i}) = \begin{pmatrix} C_{11}(\mathbf{i}) & \cdots & C_{1m}(\mathbf{i}) \\ \vdots & & \vdots \\ C_{m1}(\mathbf{i}) & \cdots & C_{mm}(\mathbf{i}) \end{pmatrix}. \square \tag{2.4}$$

In practice, the above partial derivatives often cannot be calculated directly. This is because options and other cash-flow sensitivities to interest rates typically preclude explicit mathematical modeling or the direct calculation of derivatives. In such cases, however, forward or central difference formulas can be used to estimate these derivatives, as is usually done for the traditional duration and convexity measures.

Intuitively, $D_j(\mathbf{i})$ reflects the first-order sensitivity of $P(\mathbf{i})$ to movements in the j-th yield point. For example, if $j = 10$ in the above bond yield model, changes in this yield will affect the value of cash flows at time 10 years, as well as those in the range from 8 to 19 years because of the interpolation of yields at these maturities. Similarly, $C_{jk}(\mathbf{i})$ reflects a "second-order" sensitivity of $P(\mathbf{i})$ to movements in the j-th and k-th yield points. For example, if only i_j and i_k change, $j \neq k$, $D_j(\mathbf{i})$ and $D_k(\mathbf{i})$ will reflect the first-order change in $P(\mathbf{i})$, while $C_{jk}(\mathbf{i})$ will reflect the second-order adjustment, and:

$$P(\mathbf{i} + \Delta\mathbf{i}) \approx P(\mathbf{i})\,[1 - D_j(\mathbf{i})\,\Delta i_j - D_k(\mathbf{i})\,\Delta i_k + C_{jk}(\mathbf{i})\,\Delta i_j\,\Delta i_k].$$

In matrix calculations, $\mathbf{D}(\mathbf{i})$ will be interpreted as a row matrix, while all other vectors will be identified with column vectors. Also, by the above continuity assumption on second-order partial derivatives, $C_{jk}(\mathbf{i}) = C_{kj}(\mathbf{i})$, and hence $\mathbf{C}(\mathbf{i})$ is a symmetric matrix.

Definition 2

Given $P(\mathbf{i})$ and yield curve direction vector $\mathbf{N} = (n_1, \ldots, n_m)$ with $\mathbf{N} \neq \mathbf{0}$, the *directional duration function* in the direction of $\mathbf{N}$, denoted $D_N(\mathbf{i})$, and the *directional convexity function* in the direction of $\mathbf{N}$, denoted $C_N(\mathbf{i})$, are defined for $P(\mathbf{i}) \neq 0$ as follows:

$$D_N(\mathbf{i}) = -d_N P(\mathbf{i})/P(\mathbf{i}), \tag{2.5}$$

$$C_N(\mathbf{i}) = d_N^2 P(\mathbf{i})/P(\mathbf{i}), \tag{2.6}$$

where $d_N P(\mathbf{i})$ and $d_N^2 P(\mathbf{i})$ denote the first- and second-order directional derivatives of $P(\mathbf{i})$ in the direction of $\mathbf{N}$. □

Intuitively, $\mathbf{N}$ equals the "direction" of the yield curve shift in that it reflects the relative magnitude of the individual shift amounts. A typical shift can then be modeled as $t\mathbf{N} = (tn_1, ..., tn_m)$, corresponding to each yield point i_j shifting by the amount tn_j. When all $n_j = 1$, the classical parallel shift model is produced.

A related model was developed in Ho [16], in which "key rate" durations are defined. In this context, several key rates are identified among the 360 monthly spot rates on a 30-year yield curve vector. These 360 spot rates are initially obtained using a regression model, the goal of which is to reproduce as closely as possible the price of a given collection of assets subject to certain smoothness constraints. Pyramid-type direction vectors are then defined, such as:

$$\mathbf{N}_j = (0, 0, ..1/2, 1, 2/3, 1/3, 0, ..0),$$

where the component "1" corresponds to the location of the given key rate. Also, these direction vectors form a "partition" of the parallel shift direction vector, that is:

$$\sum \mathbf{N}_j = (1, 1, ..., 1).$$

The various key rate durations are then equivalent to directional durations with the direction vectors above. For more details on the models' relationships, see Reitano [28].

The directional measures of Definition 2 can be easily obtained from the corresponding partial measures as follows:

$$D_N(\mathbf{i}) = \mathbf{D}(\mathbf{i}) \cdot \mathbf{N} = \sum n_j D_j(\mathbf{i}), \tag{2.7}$$

$$C_N(\mathbf{i}) = \mathbf{N}^T \mathbf{C}(\mathbf{i}) \mathbf{N} = \sum \sum n_j n_k C_{jk}(\mathbf{i}), \tag{2.8}$$

where $\mathbf{N}^T$ denotes the transpose of the column vector $\mathbf{N}$.

When $\mathbf{N} = (1, ..., 1)$, the associated directional measures above reduce to the more traditional modified duration and convexity measures, $D(\mathbf{i})$ and $C(\mathbf{i})$, calculated with respect to parallel yield curve shifts. In addition, we have from (2.7) and (2.8) that these traditional measures equal the sums of the corresponding partial measures:

$$D(\mathbf{i}) = \sum D_j(\mathbf{i}), \tag{2.9}$$

$$C(\mathbf{i}) = \sum\sum C_{jk}(\mathbf{i}). \tag{2.10}$$

When necessary for clarity, duration and convexity functions will explicitly reflect $P(\mathbf{i})$, such as $D_N(P)$ or $D_N(P;\mathbf{i})$ for $D_N(\mathbf{i})$.

B. Immunization Definitions

Let $P_k(\mathbf{i})$ denote the forward value of the portfolio at time $k \geq 0$, on the yield curve vector $\mathbf{i}$, where it is assumed that no securities are either added or removed from the portfolio during this period. In addition, we assume that the yield vector changes from the initial value of $\mathbf{i}_0$ to $\mathbf{i}$ immediately after time 0 and evolves according to the forward yield curve structure implied by $\mathbf{i}$ throughout the period. Letting $Z_k(\mathbf{i})$ denote the price of a k-period zero-coupon bond with maturity value of 1, it is clear that $1/Z_k(\mathbf{i})$ then equals the forward value at time k of 1 invested now. Consequently,

$$P_k(\mathbf{i}) = P(\mathbf{i})/Z_k(\mathbf{i}). \tag{2.11}$$

For example, if $i_j = i$ for all j, then $Z_k(i) = (1+i)^{-k}$ and $P_k(i) = (1+i)^k P(i)$.

Extending the classical notions of immunization, we have the following:

Definition 3

The price function $P(\mathbf{i})$ is said to be *locally immunized at time k on the yield vector* $\mathbf{i}_0$ if:

$$P_k(\mathbf{i}) \geq P_k(\mathbf{i}_0), \tag{2.12}$$

for $\mathbf{i}$ sufficiently close to $\mathbf{i}_0$; that is, for $|\mathbf{i} - \mathbf{i}_0| < r$, where $r > 0$ and $|\mathbf{i}|$ denotes the standard Euclidean norm:

$$|\mathbf{i}|^2 = \sum i_j^2. \tag{2.13}$$

Similarly, $P(\mathbf{i})$ is said to be *globally immunized at time k on the yield vector* $\mathbf{i}_0$ if (2.12) is satisfied for all feasible yield vectors $\mathbf{i}$. □

For the purposes of Definition 3, "feasibility" is not rigorously defined. Certainly, the restriction $0 < i_j$ is a minimal requirement for feasibility, though in applications other bounds may be more practical.

We analogously define *local and global immunization in the direction of* **N** by:

$$P_k(\mathbf{i}_0 + t\mathbf{N}) \geq P_k(\mathbf{i}_0) \tag{2.14}$$

for all t such that $|t|<r$ (local) and for all feasible t (global).

For the purposes of directional immunization, we restrict our attention to yield curve shifts of a fixed type, **N**, so only the amount of the shift t is variable. For example, **N** could reflect the classical parallel shift direction vector, or a shift vector that changes the yield curve level and slope, or a more general type of shift. In the nondirectional immunization model, we consider all possible directions of shift from $\mathbf{i}_0$.

Given the above definitions, we now return to the definition of $P_k(\mathbf{i})$ in (2.11) and investigate in more detail the implications of immunizing $P(\mathbf{i})$ at time k.

Assume first that $P(\mathbf{i})$ encompasses only fixed cash flows; that is:

$$P(\mathbf{i}) = \sum c_t \, [1 + r(0,t)]^{-t},$$

where $r(0,t)$ denotes the t-period spot rate, or the rate used to discount cash flows from time t to time 0. Clearly, in this notation, $Z_k(\mathbf{i}) = [1+r(0,k)]^{-k}$.

Letting $r(s,t)$ denote the rate used to discount cash flows from time t to time s, or the implied $(t-s)$-period forward rate at time s, where $0<s<t$, we have that:

$$[1 + r(0,t)]^{-t} = [1 + r(0,s)]^{-s} \, [1 + r(s,t)]^{-(t-s)}.$$

Hence, a simple calculation produces:

$$P_k(\mathbf{i}) = \sum c_t \, [1 + r(t,k)]^{(k-t)} + \sum c_t \, [1 + r(k,t)]^{-(t-k)},$$

where the first summation is over all $t<k$, and the second is over all $t \geq k$.

Consequently, we see that $P_k(\mathbf{i})$ equals the value of the cash flows at time k, where maturing cash flows are assumed to have been invested at the forward rates, and then future flows discounted at the forward rates, implied by $\mathbf{i}$. Immunization of $P(\mathbf{i})$ at time k on $\mathbf{i}_0$ then ensures that the above value will be no smaller than $P_k(\mathbf{i}_0)$, or the forward value of these cash flows based on the above formula and the forward rates implied by the current yield curve, $\mathbf{i}_0$.

It is clear from the above formula why the definition of $P_k(\mathbf{i})$ requires that the yield curve shift from $\mathbf{i}_0$ to $\mathbf{i}$ immediately after time 0 and then evolve according to the forward yield curve structure implied by $\mathbf{i}$. Otherwise, the first summation in the above formula would reflect the yield curves prevailing at the time of each maturity and reinvestment. In the current formulation, only the initial yield curve, $\mathbf{i}_0$, and the shifted yield curve, $\mathbf{i}$, have a bearing on the problem.

In the special case in which the immunization horizon k precedes the time of the first cash flow, that is, $k<t$ for all t, this assumption can be relaxed when cash flows are fixed. Specifically, in this case the yield curve shift from $\mathbf{i}_0$ to $\mathbf{i}$ can be assumed to occur at any time before time k, and can occur after any given number of other shifts; only the yield curve prevailing at time k matters. That is, immunization of $P(\mathbf{i})$ at time k on $\mathbf{i}_0$ will ensure that $P_k(\mathbf{i})$ will not fall below $P_k(\mathbf{i}_0)$ independent of the path followed by the yield curve from $\mathbf{i}_0$ to $\mathbf{i}$.

In the general case of interest-sensitive cash flows, $P_k(\mathbf{i}_0)$ in (2.11) cannot be expressed in terms of forward rates as in the above formula. However, its interpretation remains the same as the forward value of the portfolio given the current yield curve $\mathbf{i}_0$. This is because the portfolio can in theory be sold for $P(\mathbf{i}_0)$, its current market value, and the proceeds invested in a k-period zero-coupon bond, which will mature at time k for amount $P_k(\mathbf{i}_0)$.

In this context, immunization of $P(\mathbf{i})$ at time k on $\mathbf{i}_0$ again ensures that the actual forward value will not fall below this initial value when the yield curve shifts from $\mathbf{i}_0$ to $\mathbf{i}$. That is, although the current portfolio value will change from $P(\mathbf{i}_0)$ to $P(\mathbf{i})$, and the price of a zero-coupon bond will change from $Z_k(\mathbf{i}_0)$ to $Z_k(\mathbf{i})$, the above sale and purchase can still be implemented and will result in a value at time k no smaller than that originally targeted.

In contrast with the simple case of fixed cash flows, the assumption regarding the timing of the yield curve shift from $\mathbf{i}_0$ to $\mathbf{i}$, and the evolution of yields thereafter, cannot be relaxed in the case of interest-sensitive cash flows. That is, the value of the portfolio at time k will in general reflect both the timing of the shift in the simplest case, as well as the actual path the yield curve takes in the more complex case.

However, this assumption about the yield curve shift from $\mathbf{i}_0$ to $\mathbf{i}$, while necessary for the development of the theoretical results, does not prevent the application of the results to the real world. Rather, the results are fully applicable in the context of active portfolio management, whereby assets are frequently added or traded and new liabilities sold. In this context, the various criteria for immunization can be regarded as establishing targets for the asset and liability yield curve sensitivities.

For example, assume that the portfolio is structured so that surplus is immunized at time k on $\mathbf{i}_0$. Consider the value of the portfolio on the following business day, by which time the yield curve has inevitably shifted to $\mathbf{i}$. On the assumption that this shift was consistent with the direction assumed under a directional immunization program, or more generally if it is assumed that the portfolio was immunized against all shifts, it is clear that $P_k(\mathbf{i}) \geq P_k(\mathbf{i}_0)$. To remain immunized, the criteria implemented on the previous day must again be implemented in light of the new yield curve and any new assets or liabilities. In this sense, the immunization criteria become active management targets.

In the context of this day-to-day active management strategy, it makes little difference whether k is fixed in absolute terms or fixed in calendar time. In the former case, the management criteria will reflect fixed k, while in the latter case, k will be a function of time that decreases linearly as the target date approaches.

Returning to the general problem, it is clear from (2.12) and (2.14) that, for $P(\mathbf{i})$ to be immunized at time k on $\mathbf{i}_0$, $\mathbf{i}_0$ needs to be a relative minimum of $P_k(\mathbf{i})$ in the local immunization case and a global minimum in the global immunization case. For the results below, we utilize the well-known sufficient conditions for a point to be a minimum value. For example, a sufficient condition for $\mathbf{x}_0$ to be a local minimum of $f(\mathbf{x})$ in the direction of $\mathbf{N}$ is that:

$$d_N f(\mathbf{x}_0) = 0 \tag{2.15}$$

and

$$d_N^2 f(\mathbf{x}_0) > 0. \tag{2.16}$$

A sufficient condition for $\mathbf{x}_0$ to be a global minimum is that (2.15) holds and (2.16) is satisfied for all $\mathbf{x}$.

Similarly, a sufficient condition for $\mathbf{x}_0$ to be a local minimum of $f(\mathbf{x}_0)$ is that (2.15) and (2.16) hold for all $\mathbf{N}$; that is:

$$d_j f(\mathbf{x}_0) = 0, \qquad j = 1, \ldots, m, \tag{2.17}$$

and

$$(d_{jk} f(\mathbf{x}_0)) \text{ is positive definite,} \tag{2.18}$$

where $(d_{jk}f(\mathbf{x}_0))$ denotes the second derivative matrix, or Hessian matrix, of $f(\mathbf{x})$. A sufficient condition for $\mathbf{x}_0$ to be a global minimum is that (2.17) is satisfied and (2.18) holds for all $\mathbf{x}$.

We now investigate the immunization of $P(\mathbf{i})$. We will see that the durational and convexity properties of $Z_k(\mathbf{i})$ provide insight to sufficient conditions for immunization of $P(\mathbf{i})$ at time k. In particular, for local immunization we require that $P(\mathbf{i})$ have the "same duration" as $Z_k(\mathbf{i})$, and be "more convex," on the yield vector $\mathbf{i}_0$. For global immunization, we also require duration and convexity relationships on other yield curve vectors. The concepts of "same duration" and "more convex" will be made precise below, but will be seen to be natural generalizations of the classical notions to this multivariate context.

III. DIRECTIONAL IMMUNIZATION

A. General Results

This section presents general results on directional immunization. For local immunization, it is sufficient for $P(\mathbf{i})$ to have the same directional duration as $Z_k(\mathbf{i})$ and greater directional convexity.

Proposition 1

Let $P(\mathbf{i})$, $\mathbf{i}_0$ and $\mathbf{N} \neq \mathbf{0}$ be given, and assume there exists $k \geq 0$ so that on $\mathbf{i}_0$:

$$D_N(P) = D_N(Z_k), \tag{3.1}$$

and

$$C_N(P) > C_N(Z_k). \tag{3.2}$$

Then $P(\mathbf{i})$ is locally immunized at time k in the direction of $\mathbf{N}$ on the yield vector $\mathbf{i}_0$.

Proof

Applying Corollary A.4 from the Appendix to $P_k(\mathbf{i})$ in (2.11), we have from (3.1) and (3.2) that on $\mathbf{i}_0$:

$$D_N(P_k) = 0, \tag{3.3}$$

and

$$C_N(P_k) > 0. \tag{3.4}$$

Consequently, the respective directional derivatives of $P_k(\mathbf{i})$ satisfy the conditions of (2.15) and (2.16) on $\mathbf{i}_0$, since $P(\mathbf{i}_0)>0$ by assumption, and the result follows. □

For global immunization in the direction of **N**, we require a convexity constraint on all feasible yield vectors $\mathbf{i}=\mathbf{i}_0+t\mathbf{N}$.

Proposition 2

Let $P(\mathbf{i})$, $\mathbf{i}_0$ and $\mathbf{N}\neq\mathbf{0}$ be given, and assume that there exists $k\geq 0$ so that on $\mathbf{i}_0$:

$$D_N(P) = D_N(Z_k), \tag{3.5}$$

and for all feasible yield curve vectors $\mathbf{i}=\mathbf{i}_0+t\mathbf{N}$:

$$C_N(P) > C_N(Z_k) + 2D_N(Z_k)\,[D_N(P) - D_N(Z_k)]. \tag{3.6}$$

Then $P(\mathbf{i})$ is globally immunized at time k in the direction of **N** on the yield vector $\mathbf{i}_0$.

Proof

From Corollary A.4, (3.5) implies (3.3), while (3.6) implies that: (3.7)

$$C_N(P_k) > 0$$

for all feasible yield curve vectors $\mathbf{i}=\mathbf{i}_0+t\mathbf{N}$. Hence, the result follows. □

In the classical Redington model of $\mathbf{N}=(1, 1, \ldots, 1)$ and $\mathbf{i}_0$ flat, $i_j=i_0$ for all j, the conditions for local immunization reduce to familiar statements. Here, $Z_k(i_0)=v_0^k$, and using (2.7) through (2.10), Condition (3.1) becomes:

$$D(P) = k v_0, \tag{3.8}$$

from which k is uniquely determined, given i_0, by:

$$k = (1 + i_0)D(P) = D^M(P), \tag{3.9}$$

where $D^M(P)$ denotes the Macaulay duration of $P(i)$ on i_0.

Similarly, the convexity constraint in (3.2) becomes:

$$C(P) > k(k + 1)v_0^2, \tag{3.10}$$

which by (3.9) is equivalent to:

$$C(P) > D^M(P)[D^M(P) + 1]v_0^2. \tag{3.11}$$

When the cash flows underlying $P(i)$ are fixed and positive, (3.11) is always satisfied, and immunization is ensured for k satisfying (3.9). The convexity constraint in (3.10) can also be expressed in terms of the *inertia* of $P(i)$ on i_0 (see (4.10) below and Bierwag [2] for details). For more details and results from general spot rate and forward rate models, see Reitano [26].

The convexity constraints in (3.2) and (3.6) can also be expressed in terms of the directional derivatives of the directional duration functions. Specifically, because:

$$d_N D_N(P) = D_N^2(P) - C_N(P), \tag{3.12}$$

we can rewrite (3.2) as:

$$d_N D_N(P) < d_N D_N(Z_k), \tag{3.2$'$}$$

while (3.6) can be expressed:

$$d_N D_N(P) < d_N D_N(Z_k) + [D_N(P) - D_N(Z_k)]^2. \tag{3.6$'$}$$

For fixed $\mathbf{N} \neq \mathbf{0}$, the pair $(k, \mathbf{i}_0)$ of the above propositions gives rise to a *duration window* $[k, P_k(\mathbf{i}_0)]$, as defined in Bierwag [2]. Specifically, consider the graph of $y = P_x(\mathbf{i})$ in the xy-plane for each feasible $\mathbf{i} = \mathbf{i}_0 + s\mathbf{N}$. All such graphs will equal or exceed the value $P_k(\mathbf{i}_0)$ when $x = k$ in the case of global immunization, while all graphs with $|s| < r$ will have this property in the local immunization case. That is, each will pass through a window at $x = k$ with lower bound equal to $P_k(\mathbf{i}_0)$. Consequently, the value $P_k(\mathbf{i}_0)$ also gives rise to the minimum annualized return on investment over the interval $[0, k]$.

It is natural to inquire into the existence of other such duration windows. That is, given $\mathbf{i}_t = \mathbf{i}_0 + t\mathbf{N}$, does there exist $k = k(t)$ so that $P(\mathbf{i})$ is immunized at time $k(t)$ on $\mathbf{i}_t$? We next consider all such pairs, $[k(t), \mathbf{i}_t]$, and the associated duration windows, as forming an *immunization boundary*.

Definition 4

Given $P(\mathbf{i})$ and $\mathbf{N} \neq \mathbf{0}$, let $\mathbf{i}_t = \mathbf{i}_0 + t\mathbf{N}$ denote the yield vector on which $P(\mathbf{i})$ is locally (globally) immunized in the direction of $\mathbf{N}$ at time $k = k(t)$, if such a k exists. The *local (global) immunization boundary* for $P(\mathbf{i})$, in the direction of $\mathbf{N}$, denoted $IB_N(P)$, is defined:

$$IB_N(P) = \{(k, P_k(\mathbf{i}_t)) \mid k = k(t)\}. \quad \square \tag{3.13}$$

The immunization boundary then has the same property as does the duration window, yet over a range of forward times k. That is, the collection of graphs $y=P_x(\mathbf{i})$ for $\mathbf{i}=\mathbf{i}_0+s\mathbf{N}$ will be minimized at each such time $k(t)$ on the yield vector $\mathbf{i}_t$ in the global case and for more limited ranges of yield values in the local immunization case. Therefore, $P_k(\mathbf{i}_t)$ reflects the minimum portfolio value in this sense at each such time $k(t)$ and consequently gives rise to the minimum annualized return on investment, $i(k)$, over every such interval $[0,k]$:

$$i(k) = [P_k(\mathbf{i}_t)/P(\mathbf{i}_0)]^{1/k} - 1, \tag{3.14}$$

where $k=k(t)$ and $\mathbf{i}_0$ is the initial yield vector. For $t=0$, the minimum return given in (3.14) equals the annualized return on the k-period zero-coupon bond, $Z_k(\mathbf{i}_0)$, due to (2.11). Below, this return is denoted by $j(k)$.

In the classical model of $\mathbf{N}=(1, 1, ..., 1)$ and flat $\mathbf{i}_0$, the local immunization boundary always exists when cash flows are fixed and positive. This is due to the fact that, given i_t, k(t) is given by (3.9), and we have:

$$IB_N(P) = \{(k, (1 + i_t)^k P(i_t)) \mid k(t) = D^M(P, i_t)\}. \tag{3.15}$$

Consequently, the minimum return on investment in this case is given by:

$$i(k) = (1 + i_t)\ [P(i_t)/P(i_0)]^{1/k} - 1, \tag{3.16}$$

where i_0 is the initial yield value.

B. Returns on Investment: $I_k(\mathbf{i})$

As noted above, the immunization boundary gives rise to the minimum annualized return on investment, $i(k)$, over every period $[0, k]$ for which a yield vector exists so that $P(\mathbf{i})$ is immunized at time k. However, the actual return on investment over $[0, k]$ is in fact a random variable, $I_k(\mathbf{i})$, the value of which depends on the yield vector $\mathbf{i}$. As before, we assume that the initial yield vector is $\mathbf{i}_0$, that this value changes to $\mathbf{i}$ immediately after time 0, and that it evolves according to the forward yield curve structure implied by $\mathbf{i}$ throughout the period.

As in (3.14), which provides the minimum value of $I_k(\mathbf{i})$ for each k on the immunization boundary, we have for all k:

$$I_k(\mathbf{i}) = [P_k(\mathbf{i})/P(\mathbf{i}_0)]^{1/k} - 1, \tag{3.17}$$

where $\mathbf{i}=\mathbf{i}_0+t\mathbf{N}$. Following Babcock [1], we seek an approximation for $I_k(\mathbf{i})$, where the approximation reflects the dependency on t. To this end, let $\phi(t)$ denote the right-hand side of (3.17), considered as a function of t. The first-order Taylor series approximation is then $\phi(t)=\phi(0)+\phi'(0)t$.

By substitution, we have that $\phi(0)=j(k)$, where $j(k)$ is the annualized return on the zero-coupon bond, $Z_k(\mathbf{i}_0)$, as noted above. To evaluate $\phi'(0)$, note that:

$$d_t P_k(\mathbf{i})|_{t=0} = d_N P_k(\mathbf{i}_0) = -P_k(\mathbf{i}_0) D_N(P_k).$$

Consequently, we obtain the following approximation, in which all directional durations are evaluated on $\mathbf{i}_0$:

$$\begin{aligned} I_k(\mathbf{i}_0 + t\mathbf{N}) &\approx j(k) - [1 + j(k)]\, D_N(P_k)\, t/k \\ &= j(k) + [1 + j(k)]\, [D_N(Z_k) - D_N(P)] t/k. \end{aligned} \tag{3.18}$$

If $P(\mathbf{i})$ is immunized at time k, it is clear from (3.3) that the above linear approximation reduces to: $I_k(\mathbf{i}) \approx j(k)$. In this context, however, $j(k)=i(k)$, the minimum value of $I_k(\mathbf{i})$ over this period. Consequently, it is clear that the above formula is somewhat crude in this special case.

Taking the second derivative of $\phi(t)$, we obtain the following generalization of (3.18), where all durations are evaluated on $\mathbf{i}_0$:

$$\begin{aligned} I_k(\mathbf{i}_0 + t\mathbf{N}) \approx\; & j(k) - [1 + j(k)]\, D_N(P_k)\, t/k \\ & + [1 + j(k)]\, [C_N(P_k) + (k - 1)\, D_N^2\, (P_k)/k]\; t^2/2k. \end{aligned} \tag{3.19}$$

When $P(\mathbf{i})$ is immunized at time k, we see from (3.3) that the second-order bracketed term in (3.19) equals $C_N(P_k)$, which is positive by (3.4), and hence $I_k(\mathbf{i})>j(k)=i(k)$ as expected.

For more general values of k, the linear term in (3.18) and (3.19) will be non-zero. Specifically, if $P(\mathbf{i})$ is longer than $Z_k(\mathbf{i})$ on $\mathbf{i}_0$ in the direction of $\mathbf{N}$, that is, $D_N(P)>D_N(Z_k)$, then $D_N(P_k)$ will be positive and $I_k(\mathbf{i})$ will decrease with increases in the yield vector in this direction. That is, the capital loss due to the increase in yields will not be made up by reinvestment gains over the period $[0, k]$. Similarly, $I_k(\mathbf{i})$ will increase with decreases in the direction of $\mathbf{N}$. On the other hand, if $P(\mathbf{i})$ is shorter than $Z_k(\mathbf{i})$ on $\mathbf{i}_0$ in this direction, then $I_k(\mathbf{i})$ will increase with yield increases in the direction of $\mathbf{N}$, because then reinvestment gains will overcome initial capital losses.

In all cases, the second-order adjustment in (3.19) will be independent of the sign of the yield curve movement, reflecting only the magnitude of t. In general, however, the sign of this adjustment will depend on k.

Naturally, either of the above approximations can be used to estimate the mean and variance of $I_k(\mathbf{i})$, given an assumption about the probability density of t. For example, from (3.18), we obtain:

$$E[I_k(\mathbf{i}_0 + t\mathbf{N})] \approx j(k) - [1 + j(k)]\, D_N(P_k) E(t)/k, \tag{3.20}$$

$$\mathrm{Var}[I_k(\mathbf{i}_0 + t\mathbf{N})] \approx [1 + j(k)]^2\, D_N^2(P_k)\ \mathrm{Var}(t)/k^2. \tag{3.21}$$

IV. ASSET/LIABILITY MANAGEMENT

In this section, we translate the above immunization methodology and results to an asset/liability management setting. To this end, we consider two objective functions:

$$S(\mathbf{i}) = A(\mathbf{i}) - L(\mathbf{i}), \tag{4.1}$$

$$R(\mathbf{i}) = [A(\mathbf{i}) - L(\mathbf{i})]/A(\mathbf{i}), \tag{4.2}$$

where $A(\mathbf{i})$ and $L(\mathbf{i})$ denote the market values of assets and liabilities, respectively. Immunization in the context of (4.1) then provides a floor for the value of surplus at time k, while use of the objective function in (4.2) provides a floor for the ratio of surplus to assets, or net worth asset ratio, or simply, surplus ratio.

A. Immunization of Surplus

Let r^s denote the surplus ratio on the current yield vector $\mathbf{i}_0$; that is,

$$r^s = R(\mathbf{i}_0) = [A(\mathbf{i}_0) - L(\mathbf{i}_0)]/A(\mathbf{i}_0). \tag{4.3}$$

Proposition 3

Let $S(\mathbf{i}) = A(\mathbf{i}) - L(\mathbf{i})$, $\mathbf{i}_0$ and $\mathbf{N} \neq \mathbf{0}$ be given. Assume that there exists $k \geq 0$ so that on $\mathbf{i}_0$:

$$D_N(A) = (1 - r^s)D_N(L) + r^s D_N(Z_k), \tag{4.4}$$

$$C_N(A) > (1 - r^s)C_N(L) + r^s C_N(Z_k). \tag{4.5}$$

Then $S(\mathbf{i})$ is locally immunized at time k in the direction of $\mathbf{N}$ on the yield vector $\mathbf{i}_0$.

Proof

Consider first the case in which $r^s > 0$. By Proposition 1, we require on $\mathbf{i}_0$:

$$D_N(S) = D_N(Z_k). \tag{4.6}$$

However, by Corollary A.1,

$$D_N(S) = D_N(A)/r^s - D_N(L)(1 - r^s)/r^s,$$

and (4.6) follows from (4.4). An identical argument demonstrates that (4.5) is equivalent to $C_N(S) > C_N(Z_k)$ on $\mathbf{i}_0$.

For the case $r^s = 0$, we work directly with the directional derivatives of $S_k(\mathbf{i})$, with the goal that (2.15) and (2.16) be satisfied. The resulting conditions on the directional derivatives of $A(\mathbf{i})$ and $L(\mathbf{i})$ are then equivalent to the conditions in (4.4) and (4.5) with $r^s = 0$. □

When $r^s = 0$, Conditions (4.4) and (4.5) imply that $S(\mathbf{i})$ is locally immunized at all times $k \geq 0$ in the direction of $\mathbf{N}$ on the yield vector $\mathbf{i}_0$. Consequently, the local immunization boundary is given by (3.13) with $\mathbf{i}_t = \mathbf{i}_0$ for all $k \geq 0$. However, since $r^s = 0$, we have that $S_k(\mathbf{i}_0) = 0$ for all k, and hence,

$$IB_N(S) = \{(k, 0) : k \geq 0\}. \tag{4.7}$$

For $r^s > 0$, we see that the directional duration of assets required for immunization reflects both the directional durations of liabilities and the zero-coupon bond, $Z_k(\mathbf{i})$, corresponding to the immunization horizon k. In some applications, k may be chosen small or equal to zero, providing short-term immunization as part of an active management strategy.

For $k = 0$, the above conditions become:

$$D_N(A) = (1 - r^s)D_N(L), \tag{4.8}$$

$$C_N(A) > (1 - r^s)C_N(L). \tag{4.9}$$

When $\mathbf{N} = (1, \ldots, 1)$, the parallel shift direction vector, and the yield curve $\mathbf{i}_0$ is flat, the above conditions are equivalent to those in Bierwag [2], stated in terms of Macaulay durations and the portfolio inertias I_A. This is because in this case:

$$(1 + i)^2 C(A) = I_A + D^M(D^M - 1), \tag{4.10}$$

and similarly for liabilities. In this special case, it is clear from (4.4) and (4.5) that immunization at time $k > 0$ requires more asset duration and convexity as k increases, because then $D_N(Z_k) = kv$ is an increasing function of k, as is $C_N(Z_k) = k(k+1)v^2$.

More generally, k can be chosen to be consistent with the planning cycle of the organization. For example, $k = 1$ would be an initial immunization target consistent with stabilizing income over a one-period interval, where income is defined as the change in net worth. In such a strategy, the value of k would be decreased over the period consistent with the targeting of values to a fixed calendar date, such as December 31. Similarly, larger values of k can be chosen to reflect a multiyear business plan or the maturity period of the last liability flow. This last assignment would then be consistent with immunizing pricing margins over the life of a block of liabilities.

As noted in Section IIB, however, the assumption that $\mathbf{i}_0$ shifts to $\mathbf{i}$ immediately after time 0 and remains fixed during $[0, k]$ effectively precludes the use of the above results as part of a passive management strategy, that is, a strategy whereby the portfolio is structured at time 0 and effectively left alone during the period, except perhaps for the reinvestment of maturing cash flows. As noted there, passive management is possible in theory in the special case of fixed cash flows if the planning horizon, k, is less than the time of the first cash flow. However, even in this case immunization could fail under a local immunization strategy if the yield curve shift during the period is too great.

B. Immunization of the Surplus Ratio

In this section, we investigate the immunization of the net worth asset ratio, $R(\mathbf{i}) = [A(\mathbf{i}) - L(\mathbf{i})]/A(\mathbf{i})$. Since $R(\mathbf{i})$ is not a price function, its forward value at time k, $R_k(\mathbf{i})$, is not given by (2.11). However, we have:

$$\begin{aligned} R_k(\mathbf{i}) &= [A_k(\mathbf{i}) - L_k(\mathbf{i})]/A_k(\mathbf{i}) \\ &= R(\mathbf{i}), \end{aligned}$$

because the forward values of $A(\mathbf{i})$ and $L(\mathbf{i})$ satisfy (2.11). Consequently, immunizing $R(\mathbf{i})$ at time 0 ensures its immunization at all times $k \geq 0$.

Proposition 4

Let $R(\mathbf{i})$ be defined as above, and let $\mathbf{i}_0$ and $\mathbf{N} \neq \mathbf{0}$ be given. Assume that on $\mathbf{i}_0$:

$$D_N(A) = D_N(L), \tag{4.11}$$

$$C_N(A) > C_N(L). \tag{4.12}$$

Then $R(\mathbf{i})$ is locally immunized at all times $k \geq 0$ in the direction of $\mathbf{N}$ on the yield vector $\mathbf{i}_0$.

Proof

Assume that $R(\mathbf{i}_0) = r^s > 0$. We then have from Corollaries A.4 and A.1:

$$D_N(R) = D_N(A - L) - D_N(A)$$
$$= c[D_N(A) - D_N(L)],$$

where $c = L(\mathbf{i}_0)/S(\mathbf{i}_0)$. Consequently, (2.15) is satisfied due to (4.11). Similarly:

$$C_N(R) = c[C_N(A) - C_N(L)] - 2cD_N(A)\,[D_N(A) - D_N(L)],$$

and (2.16) is satisfied due to (4.11) and (4.12).

For $r^s = 0$, we proceed as in Proposition 3, working directly with the directional derivatives of $R(\mathbf{i})$. □

When $\mathbf{N} = (1, \ldots, 1)$ and the yield vector $\mathbf{i}_0$ is flat, the above conditions reduce to those in Bierwag [2] expressed in terms of Macaulay durations and inertias due to (4.10). Also, for general $\mathbf{N}$, the local immunization boundary in (3.13) is given with $\mathbf{i}_t = \mathbf{i}_0$ for all $k \geq 0$, and hence, $R_k(\mathbf{i}_t) = r^s$; that is,

$$IB_N(R) = \{(k, r^s) \mid k \geq 0\}. \tag{4.13}$$

We leave it to the reader to generalize Propositions 3 and 4 to the case of global immunization in the direction of $\mathbf{N}$.

C. An Example

The significance of Propositions 3 and 4 is the theoretical dependence of the condition of immunization on the direction vector, $\mathbf{N}$, assumed in the duration and convexity estimates. In the classical model, $\mathbf{N} = (1, 1, \ldots, 1)$ is commonly assumed.

In theory, the satisfaction of Conditions (4.4) and (4.5), or (4.11) and (4.12) for a given $\mathbf{N}$, does not ensure their satisfaction for other values of $\mathbf{N}$. The following example illustrates that this observation is also true in practice and demonstrates how surplus immunization may fail due to actual yield curve shifts not encompassed by the model's specification for $\mathbf{N}$. For other examples that relate to surplus and the net worth asset ratio and illustrate the same conclusion, see Reitano [29].

Assume that the yield curve is given by $\mathbf{i}_0 = (0.075, 0.090, 0.100)$, representing bond yields at time 0.5, 5, and 10 years, respectively. For all valuations below, we assume that bond yields at other maturities are derived by linear interpolation and that spot yields are developed from these values by the usual procedure. That is, they are derived so as to price the bonds implied by the bond yield curve to par. In practice, more pivotal points or yield curve drivers would be used in the valuation model. For example, maturities of 1, 3, and 7 years would often be added to the bond yield curve, but we use this model for simplicity.

Assets are to be composed of a mix of a 10-year, 12 percent coupon bond, and a 6-month pure discount position, such as commercial paper. Based on the above yield curve, this bond has a market value of 112.80 per 100 of par and a duration of 6.151, while the commercial paper has a market value of 96.39 per 100 and a duration of 0.482. To be consistent with the partial duration basis below, these duration values reflect sensitivity to parallel shifts in the bond yield curve and were calculated by using a forward difference approximation to the derivative with a difference of 5 b.p. (see Reitano [24, 25, 28] for more detail).

The liability is a \$100-million 5-year zero-coupon bond, such as a 5-year guaranteed investment contract (GIC), with a current market value of \$63.97 million and duration of 4.855. Available assets total \$71.08 million, and hence $S(\mathbf{i}_0)$ = \$7.11 million and $r^s = 0.10$ for a 10 percent net worth asset ratio.

We seek to apply Proposition 3 to immunize surplus from parallel shifts at time $k=1/2$, the time of the first cash flow. First, the duration of assets must satisfy (4.4) and hence must equal 4.418. A calculation based on Proposition A.1 shows that about 31 percent of assets needs to be invested in commercial paper, purchasing a par value of \$22.54 million, while the remainder is to be invested in the bond, purchasing \$43.75 million par. The asset portfolio then has a duration of 4.418, producing a surplus duration of 0.482, the same as the duration of $Z_k(\mathbf{i}_0)$ for $k=1/2$. Consequently, (4.6) and hence (4.4) are satisfied. In addition, because the convexity of the bond, commercial paper, and GIC equal 52.48, 0.46, and 25.95, respectively, the convexity of surplus equals 132.25. Since this value exceeds the convexity of $Z_k(\mathbf{i}_0)$ of 0.46, (4.5) is also satisfied.

Consequently, Proposition 3 assures that this portfolio is locally immunized at time $k=1/2$ against parallel shifts from the initial yield vector $\mathbf{i}_0=(0.075, 0.090, 0.100)$. For $k=1/2$, the forward value of surplus, $S_k(\mathbf{i}_0)$, equals \$7.37 million, which then provides the minimum value of $S_k(\mathbf{i})$, for $\mathbf{i}=\mathbf{i}_0+t\mathbf{N}$, $\mathbf{N}=(1, 1, 1)$, and small t. In addition, by (3.14) the minimum annualized half-year return on surplus, $i(k)$, is the return on $Z_k(\mathbf{i}_0)$ of 7.64 percent, the annual equivalent of 7.50 percent.

Using (3.19), we estimate the actual half-year return on surplus for $\mathbf{N}=(1, 1, 1)$:

$$I_k(\mathbf{i}_0 + t\mathbf{N}) \approx 0.0764 + 1.0764(131.77t^2), \quad k = 1/2, \tag{4.14}$$

while for $S_k(\mathbf{i})$, the corresponding Taylor series is:

$$S_k(\mathbf{i}_0 + t\mathbf{N}) \approx 7.37[1 + \tfrac{1}{2} \cdot 131.77t^2]. \tag{4.15}$$

A calculation then produces the following results:

t	$S_k(\mathbf{i})$	$S_k^e(\mathbf{i})$	$I_k(\mathbf{i})$	$I_k^e(\mathbf{i})$
−0.02	7.59	7.57	14.1%	13.3%
−0.01	7.43	7.42	9.2	9.1
−0.005	7.39	7.39	8.0	8.0
0	7.37	7.37	7.6	7.6
0.005	7.38	7.39	8.0	8.0
0.01	7.42	7.42	8.9	9.1
0.02	7.55	7.57	12.7	13.3

In this table, S_k and I_k represent exact values, S_k^e and I_k^e estimates using (4.14) and (4.15). The above values demonstrate that immunization at time $k=1/2$ will be successful if yield curve shifts are parallel. In addition, it illustrates the degree of accuracy of the above approximations in this case.

For nonparallel shifts $\mathbf{N}$, the conclusions can be significantly different. Consider first the duration value. By construction, $D_N(S_k) = D_N(S) - D_N(Z_k) = 0$ for $\mathbf{N} = (1, 1, 1)$. However, calculating the partial durations of $S_k(\mathbf{i}_0)$ and using (2.7), we have that in general on $\mathbf{i}_0$, for $N = (n_1, n_2, n_3)$:

$$D_N(S_k) = 5.26n_1 - 46.21n_2 + 40.95n_3. \tag{4.16}$$

To evaluate the potential range of directional durations in (4.16), a restriction must first be put on the length of $\mathbf{N}$ because $D_N(S_k)$ is proportional to $|\mathbf{N}|$. Because we seek to compare the resulting values to that produced by $\mathbf{N} = (1, 1, 1)$, which has a length of $\sqrt{3}$, we restrict $|\mathbf{N}|$ to equal $\sqrt{3}$ for consistency. A calculation then produces (see Reitano [24, 28]):

$$-107.33 \le D_N(S_k) \le 107.33, \quad |\mathbf{N}| = \sqrt{3}. \tag{4.17}$$

The boundary points in (4.17) equal $\pm\sqrt{3}|\mathbf{D}(S_k)|$ and are achieved when $\mathbf{N}$ is proportional to $\mathbf{D}(S_k)$. For example, a simple calculation shows that $\mathbf{N} = (0.147, -1.292, 1.145)$ has length $\sqrt{3}$ (approximately), equals 2.8 percent of $\mathbf{D}(S_k)$, and produces $D_N(S_k) \approx 107.33$.

Consequently, while $D_N(S_k) = 0$ for $\mathbf{N} = (1, 1, 1)$, nonparallel shifts expose this portfolio to significant duration risk. To analyze convexity, we require the total convexity matrix $\mathbf{C}(S_k)$. When $D_N(S_k) = 0$, $C_N(S_k) = C_N(S) - C_N(Z_k)$

by Corollary A.4. In general, however, we must include the duration terms, producing:

$$\begin{aligned} C_N(S_k) = {} & 9.03n_1^2 - 162.73n_2^2 + 167.76n_3^2 \\ & - 67.10n_1n_2 + 25.72n_1n_3 + 159.10n_2n_3 \\ & - 0.9636n_1 D_N(S_k). \end{aligned} \tag{4.18}$$

As noted above, $C_N(S_k) = C_N(S) - C_N(Z_k) = 131.77$ when $\mathbf{N} = (1, 1, 1)$. For other $\mathbf{N}$ with $|\mathbf{N}| = \sqrt{3}$, we use the result that (see Reitano [28]):

$$3\lambda^m \leq C_N(S_k) \leq 3\lambda^M, \qquad |\mathbf{N}| = \sqrt{3} \tag{4.19}$$

where λ^m, λ^M represent the minimum and maximum eigenvalues of the total convexity matrix, $\mathbf{C}(S_k)$, given in Proposition A.4:

$$\mathbf{C}(S_k) = \begin{pmatrix} 3.97 & -11.29 & -6.87 \\ -11.29 & -162.73 & 79.55 \\ -6.87 & 79.55 & 167.76 \end{pmatrix}. \tag{4.20}$$

A calculation then produces eigenvalues of -181.4, 4.0 and 186.4, and (4.19) becomes:

$$-544.2 \leq C_N(S_k) \leq 559.2. \tag{4.21}$$

Consequently, the estimate $C_N(S_k) = 131.77$ for $\mathbf{N} = (1, 1, 1)$ understates the potential magnitude of the convexity factor. More importantly, it disguises its potential sign, because it is often tacitly assumed that the convexity adjustment for such a portfolio is always positive.

By utilizing monthly Treasury yield data for the period August 1984 to June 1990, 65 sample values were produced for yield change vectors, $\mathbf{N}$, representing overlapping 6-month yield curve shifts.

Actual values of $D_N(S_k)$ using (4.16) ranged from -0.153 to 0.148, while actual values of $C_N(S_k)$ using (4.18) ranged from -0.007 to 0.148. Normalizing all values of $\mathbf{N}$ so that $|\mathbf{N}| = \sqrt{3}$, we obtained the following values for this period:

$$\begin{aligned} -15.75 \leq D_N(S_k) \leq 40.17, \\ -217.11 \leq C_N(S_k) \leq 447.51. \end{aligned} \tag{4.22}$$

Consequently, this sample period produced 6-month yield curve shifts that demonstrated significantly different normalized values of D_N and C_N compared with the respective values for $\mathbf{N} = (1, 1, 1)$ of 0 and 131.77. These observed ranges can be compared to the theoretical ranges in (4.17) and (4.21). In the case of C_N, values close to the theoretical maximum were observed. See Table 1 for the distribution of results.

TABLE 1

DISTRIBUTION OF DIRECTIONAL DURATIONS AND CONVEXITIES
PERCENTILES FOR 65 OVERLAPPING 6-MONTH PERIODS
AUGUST 1984 TO JUNE 1990

	Actual		Normalized $\|N\| = \sqrt{3}$	
Percentile	$D_N(S_k)$	$C_N(S_k)$	$D_N(S_k)$	$C_N(S_k)$
0.02	−0.153	−0.007	−15.75	−217.11
0.10	−0.105	−0.003	−11.20	−48.89
0.20	−0.069	−0.001	−7.67	−8.79
0.30	−0.056	0.001	−5.13	30.87
0.40	−0.035	0.003	−3.27	62.96
0.48*			0	
0.50	0.009	0.006	1.91	90.59
0.59*				131.77
0.60	0.034	0.008	4.86	139.96
0.70	0.048	0.014	5.73	188.48
0.80	0.068	0.021	7.31	238.98
0.90	0.094	0.043	11.60	283.49
1.00	0.148	0.148	40.17	447.51

*Values for **N** = (1, 1, 1).

Utilizing the actual values of **N**, $S_k(\mathbf{i})$ and $I_k(\mathbf{i})$ can be estimated using the calculated duration and convexity values. For this purpose, we use (3.19) and the following generalization of (4.15):

$$S_k(\mathbf{i}_0 + t\mathbf{N}) \approx S_k(\mathbf{i}_0)\,[1 - D_N(S_k)t + \tfrac{1}{2}C_N(S_k)t^2]. \tag{4.23}$$

The ranges produced were:

$$\begin{aligned} 6.32 &\leq S_k^e(\mathbf{i}) \leq 8.68, \\ -0.208 &\leq I_k^e(\mathbf{i}) \leq 0.482. \end{aligned} \tag{4.24}$$

Table 2 displays the distribution of estimated results and shows that immunization was unsuccessful in more than 50 percent of the periods observed. Table 3 displays a comparison of actual and estimated values over 11 nonoverlapping 6-month periods and shows immunization failing in 6 of the 11 periods. Note the proximity of actual and estimated values on Table 3, indicating the extent to which this portfolio's risk characteristics were captured by $\mathbf{D}(S_k)$ and $\mathbf{C}(S_k)$.

TABLE 2

DISTRIBUTION OF ESTIMATED PERIOD RETURNS
AND PERIOD-END VALUES
PERCENTILES FOR 65 OVERLAPPING
6-MONTH PERIODS
AUGUST 1984 TO JUNE 1990

Percentile	$I_k^e(\mathbf{i})$	$S_k^e(\mathbf{i})$
0.02	−20.818%	$6.3182
0.10	−11.853	6.6701
0.20	−7.536	6.8349
0.30	−1.971	7.0357
0.40	1.156	7.1470
0.50	4.869	7.2778
0.53*	7.641	7.3743
0.60	14.035	7.5908
0.70	22.025	7.8523
0.80	29.105	8.0772
0.90	33.165	8.2126
1.00	48.190	8.6773

*Expected values on initial yield curve, $\mathbf{i}_0$ = (0.075, 0.09, 0.10).

TABLE 3

ACTUAL VERSUS ESTIMATED VALUES
NONOVERLAPPING 6-MONTH PERIODS
AUGUST 1984 TO JUNE 1990

6 mos. beginning	$S_k(\mathbf{i})$	$S_k^e(\mathbf{i})$	$I_k(\mathbf{i})$	$I_k^e(\mathbf{i})$
1/1/85*	$6.9558	$6.9471	−4.230%	−4.337%
7/1/85*	7.2915	7.2834	5.239	5.056
1/1/86	8.6943	8.6773	49.625	48.190
7/1/86*	6.4331	6.4340	−18.084	−18.089
1/1/87	7.9502	7.9589	25.110	25.227
7/1/87*	7.1163	7.1178	0.240	0.299
1/1/88*	7.3425	7.3421	6.714	6.704
7/1/88	8.1939	8.1936	32.899	32.951
1/1/89*	7.0121	7.0064	−2.674	−2.744
7/1/89	7.4207	7.4201	8.999	8.981
1/1/90	7.5890	7.5908	13.999	14.036

*Immunization unsuccessful: $S_k(\mathbf{i}_0)$ = $7.37, $I_k(\mathbf{i}_0)$ = 7.64%.

Recall that this portfolio was immunized against parallel shifts, with expected minimums: $S_k(\mathbf{i}_0)=7.37$, $I_k(\mathbf{i}_0)=0.076$. Consequently, parallel shift immunization assured immunization against nonparallel shifts neither in theory (Proposition 3) nor in practice (Tables 2 and 3). However, the above methodology provides a framework for measuring potential risk, as well as insight to conditions under which complete immunization would be assured.

V. NONDIRECTIONAL IMMUNIZATION

A. General Results

In this section, general results on nondirectional immunization are developed and seen to be natural generalizations of the Section III results. For local immunization, for example, we again require $P(\mathbf{i})$ to have the "same duration" as $Z_k(\mathbf{i})$ on $\mathbf{i}_0$ and be "more convex." Here, however, the constraints are stated in terms of the total duration vectors and total convexity matrices. We begin with a definition:

Definition 5

Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices. We say that $\mathbf{A}$ *is more convex than* $\mathbf{B}$, denoted $\mathbf{A} > \mathbf{B}$, if $\mathbf{A} - \mathbf{B}$ is positive definite. That is, $\mathbf{x}^T(\mathbf{A} - \mathbf{B})\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. □

For convenience, we will sometimes write $\mathbf{A} > \mathbf{0}$, which by Definition 5 means that $\mathbf{A}$ is positive definite.

The generalization of Proposition 1 is then:

Proposition 5

Let $P(\mathbf{i})$ and $\mathbf{i}_0$ be given and assume that there exists a $k \geq 0$ so that on $\mathbf{i}_0$:

$$\mathbf{D}(P) = \mathbf{D}(Z_k), \tag{5.1}$$

$$\mathbf{C}(P) > \mathbf{C}(Z_k). \tag{5.2}$$

Then $P(\mathbf{i})$ is locally immunized at time k on the yield vector $\mathbf{i}_0$.

Proof

As for the proof of Proposition 1, we require the result of Proposition A.4 relating $\mathbf{D}$ and $\mathbf{C}$ for $P_k(\mathbf{i})$ to the respective values for $P(\mathbf{i})$ and $Z_k(\mathbf{i})$. In particular, from (A.13) we see that (5.1) assures that:

$$\mathbf{D}(P_k) = \mathbf{0}, \tag{5.3}$$

while (5.2) and (5.1) together imply that:

$$\mathbf{C}(P_k) > \mathbf{0}. \tag{5.4}$$

Recalling Conditions (2.17) and (2.18), we see that the above conclusions regarding $P_k(\mathbf{i})$ ensure that $\mathbf{i}_0$ is a local minimum, and the result follows. □

Clearly, the conditions of Proposition 5 are equivalent to assuming that Conditions (3.1) and (3.2) of Proposition 1 are satisfied for a fixed k, for all direction vectors $\mathbf{N}$. A similar statement holds for the generalization of Proposition 2, which we state without proof.

Proposition 6

Let $P(\mathbf{i})$ and $\mathbf{i}_0$ be given and assume that there exists a $k \geq 0$ so that on $\mathbf{i}_0$:

$$\mathbf{D}(P) = \mathbf{D}(Z_k), \tag{5.5}$$

and for all feasible $\mathbf{i}$:

$$\mathbf{C}(P) - \mathbf{C}(Z_k) > 2\mathbf{D}(Z_k)^T [\mathbf{D}(P) - \mathbf{D}(Z_k)]. \tag{5.6}$$

Then $P(\mathbf{i})$ is globally immunized at time k on the yield vector $\mathbf{i}_0$. □

B. Returns on Investment: $\mathrm{I}_k(\mathbf{i})$

Defining $I_k(\mathbf{i})$ as in (3.17), we have the following counterpart to (3.18), which follows from (2.7):

$$I_k(\mathbf{i}) \approx j(k) - [1 + j(k)]\, \mathbf{D}(P_k)\cdot(\mathbf{i} - \mathbf{i}_0)/k. \tag{5.7}$$

The second-order term in (3.19) can be similarly expressed.

The earlier comments about the competition between capital gains and losses and reinvestment losses and gains apply here as well. Here, however, the concept of $P(\mathbf{i})$ being "longer" or "shorter" than $Z_k(\mathbf{i})$ refers to the sign of the inner product in (5.7) being positive or negative, respectively.

To generalize the moments of $I_k(\mathbf{i})$ in (3.20) and (3.21), we require the following notation. Let $\mathbf{E}(\mathbf{i}-\mathbf{i}_0)$ denote the vector mean, and $\mathbf{V}(\mathbf{i}-\mathbf{i}_0)$ denote the covariance matrix of $\mathbf{i}-\mathbf{i}_0$, reflecting the underlying density function of $\mathbf{i}$. Then:

$$E[I_k(\mathbf{i})] \approx j(k) - [1 + j(k)]\, \mathbf{D}(P_k)\cdot\mathbf{E}(\mathbf{i} - \mathbf{i}_0)/k, \tag{5.8}$$

$$\mathrm{Var}[I_k(\mathbf{i})] \approx [1 + j(k)]^2\, \mathbf{D}(P_k)\, \mathbf{V}(\mathbf{i} - \mathbf{i}_0)\, \mathbf{D}(P_k)^T/k^2, \tag{5.9}$$

where all total duration vectors are evaluated on $\mathbf{i}_0$.

C. Asset-Liability Management

For nondirectional immunization, the results of Section IV generalize in the natural way. We state the results without proof.

Proposition 7

Let $S(\mathbf{i}) = A(\mathbf{i}) - L(\mathbf{i})$ and $\mathbf{i}_0$ be given. Assume that there exists $k \geq 0$, so that on $\mathbf{i}_0$:

$$\mathbf{D}(A) = (1 - r^s)\mathbf{D}(L) + r^s\mathbf{D}(Z_k), \tag{5.10}$$

$$\mathbf{C}(A) > (1 - r^s)\mathbf{C}(L) + r^s\mathbf{C}(Z_k). \tag{5.11}$$

Then $S(\mathbf{i})$ is locally immunized at time k on the yield vector $\mathbf{i}_0$. □

As for Proposition 3, the conclusion of Proposition 7 remains valid when $r^s = 0$. Conditions (5.10) and (5.11) then imply local immunization for all $k \geq 0$. Also, in the same way that (4.4) of Proposition 3 implies that $D_N(S_k) = 0$, (5.10) of Proposition 7 is equivalent to $\mathbf{D}(S_k) = \mathbf{0}$.

Returning to the above example, the durational constraint imposed by (5.10) to ensure complete immunization is that $\mathbf{D}(A) = (-0.354, 4.772, 0)$. This total duration vector is substantially different from that of the given assets of $\mathbf{D}(A) = (0.172, 0.152, 4.095)$.

Proposition 8

Let $R(\mathbf{i})$ be defined as in (3.16) and $\mathbf{i}_0$ be given. Assume that on $\mathbf{i}_0$:

$$\mathbf{D}(A) = \mathbf{D}(L), \tag{5.12}$$

$$\mathbf{C}(A) > \mathbf{C}(L). \tag{5.13}$$

Then $R(\mathbf{i})$ is locally immunized at all times $k \geq 0$ on the yield vector $\mathbf{i}_0$. □

VI. YIELD VECTOR TRANSFORMATIONS

It is natural to inquire to what extent immunization, as developed above, depends on the underlying yield vector basis used. For example, if a portfolio is locally immunized at time k on the yield vector $\mathbf{i}_0$, what can be said if the analysis was to be done using yield basis $\mathbf{j}_0$? A similar question arises for directional immunization. The next proposition shows that the property of local immunization is independent of the yield basis.

Here and throughout this section the yield curve basis is displayed as part of the duration and convexity notation to avoid confusion.

Proposition 9

Let $P(\mathbf{i})$ be a price function that satisfies Conditions (5.1) and (5.2) of Proposition 5 and hence is locally immunized at time k on the yield vector

$\mathbf{i}_0$. Let $\mathbf{A}$:$\mathbf{i}\rightarrow\mathbf{j}$ be a yield curve transformation, with a nonsingular Jacobian matrix, $\mathbf{J}[\mathbf{A}(\mathbf{i})]$ at $\mathbf{i}_0$. Then $P(\mathbf{j}) \equiv P(\mathbf{A}^{-1}(\mathbf{j}))$ also satisfies these conditions on $\mathbf{j}_0 = \mathbf{A}(\mathbf{i}_0)$ and hence is also locally immunized at time k on the yield vector $\mathbf{j}_0$.

Proof

By Proposition A.5, we have:

$$\mathbf{D}(P_k; \mathbf{i}_0) = \mathbf{D}(P_k; \mathbf{j}_0)\, \mathbf{J}[\mathbf{A}(\mathbf{i}_0)],$$

and hence:

$$[\mathbf{D}(P; \mathbf{i}_0) - \mathbf{D}(Z_k; \mathbf{i}_0)] = [\mathbf{D}(P; \mathbf{j}_0) - \mathbf{D}(Z_k; \mathbf{j}_0)]\, \mathbf{J}[\mathbf{A}(\mathbf{i}_0)]. \quad (6.1)$$

Consequently, since $\mathbf{J}[\mathbf{A}(\mathbf{i}_0)]$ is nonsingular, $P(\mathbf{i})$ satisfies (5.1) on $\mathbf{i}_0$ if and only if it satisfies this constraint on $\mathbf{j}_0$.

Similarly, we have:

$$\mathbf{C}(P_k; \mathbf{i}_0) = \mathbf{J}[\mathbf{A}(\mathbf{i}_0)]^T\, \mathbf{C}(P_k; \mathbf{j}_0)\, \mathbf{J}[\mathbf{A}(\mathbf{i}_0)] - \mathbf{D}(P_k; \mathbf{j}_0)\, \mathbf{H}[\mathbf{A}(\mathbf{i}_0)],$$

where $\mathbf{H}[\mathbf{A}(\mathbf{i}_0)]$ is the Hessian "matrix" of $\mathbf{A}$ at $\mathbf{i}_0$. Substituting for the total convexity matrixes using (A.14) and using the fact $\mathbf{D}(P_k; \mathbf{i}_0) = \mathbf{D}(P_k; \mathbf{j}_0) = \mathbf{0}$ by (6.1), we obtain:

$$\mathbf{C}(P; \mathbf{i}_0) - \mathbf{C}(Z_k; \mathbf{i}_0) = \mathbf{J}[\mathbf{A}(\mathbf{i}_0)]^T\, [\mathbf{C}(P; \mathbf{j}_0) - \mathbf{C}(Z_k; \mathbf{j}_0)]\, \mathbf{J}[\mathbf{A}(\mathbf{i}_0)]. \quad (6.2)$$

Consequently, since $\mathbf{J}[\mathbf{A}(\mathbf{i}_0)]$ is nonsingular, $\mathbf{C}(P)$ satisfies (5.2) on $\mathbf{i}_0$ if and only if it satisfies this constraint on $\mathbf{j}_0$. □

The implication of Proposition 9 is clear, namely, that k, the time to which $P(\mathbf{i})$ is immunized, is an invariant and intrinsic property of the portfolio. It does not depend on the yield curve basis chosen. For directional immunization, the situation is of necessity more yield curve dependent, because the direction vector $\mathbf{N}$ clearly reflects the yield curve basis. Transforming $\mathbf{N}$ by the Jacobian of the transformation provides a direction vector $\mathbf{M}$ for which immunization is possible, yet unfortunately not assured without additional constraints, as the following result demonstrates.

Proposition 10

Let $P(\mathbf{i})$ be a price function and $\mathbf{N} \neq \mathbf{0}$ a direction vector such that Conditions (3.1) and (3.2) of Proposition 1 are satisfied, and hence $P(\mathbf{i})$ is locally immunized at time k in the direction of $\mathbf{N}$ on the yield vector $\mathbf{i}_0$. Let $\mathbf{A}$ be given as above. Then $P(\mathbf{j})$ satisfies Condition (3.1) with $\mathbf{M}=\mathbf{J}[\mathbf{A}(\mathbf{i}_0)]\mathbf{N}$ and $\mathbf{j}_0=\mathbf{A}(\mathbf{i}_0)$. In addition, if $D_{\mathbf{M}'}(P;\mathbf{j}_0)\geq D_{\mathbf{M}'}(Z_k;\mathbf{j}_0)$, where $\mathbf{M}'=\mathbf{N}^T\mathbf{H}[\mathbf{A}(\mathbf{i}_0)]\mathbf{N}$, then $P(\mathbf{j})$ also satisfies Condition (3.2) and hence is also locally immunized at time k in the direction of $\mathbf{M}$ on the yield vector $\mathbf{j}_0$.

Proof

Using Corollary A.5, we have:

$$D_N(P_k;\mathbf{i}_0) = D_M(P_k;\mathbf{j}_0), \tag{6.3}$$

and hence $P(\mathbf{j})$ satisfies Condition (3.1) with $\mathbf{M}$ and $\mathbf{j}_0$ if and only if $P(\mathbf{i})$ satisfies this condition with $\mathbf{N}$ and $\mathbf{i}_0$.

Using the corresponding result for directional convexities, and simplifying, we obtain:

$$C_N(P_k;\mathbf{i}_0) - C_N(Z_k;\mathbf{i}_0) = C_M(P_k;\mathbf{j}_0) - C_M(Z_k;\mathbf{j}_0) - D_{M'}(P_k;\mathbf{j}_0). \tag{6.4}$$

Consequently, if $P(\mathbf{i})$ satisfies (3.2) with $\mathbf{N}$ and $\mathbf{i}_0$, it does not necessarily follow that $P(\mathbf{j})$ satisfies this condition with $\mathbf{M}$ and $\mathbf{j}_0$ due to the last term on the right of (6.4). However, if $D_{M'}(P_k;\mathbf{j}_0)=D_{\mathbf{M}'}(P;\mathbf{j}_0)-D_{M'}(Z_k;\mathbf{j}_0)\geq 0$, local immunization in the direction of $\mathbf{M}$ is ensured. □

Results about global immunization can be treated similarly. Unfortunately, as in Proposition 10, while the duration results carry forward well, the convexity conditions are not preserved without additional constraints. For example, for global immunization, we require $\mathbf{J}[\mathbf{A}(\mathbf{i})]$ to be nonsingular everywhere and $\mathbf{D}(P_k;\mathbf{j})\mathbf{H}[\mathbf{A}(\mathbf{i})]$ to be positive definite for all $\mathbf{i}$. Details are left to the interested reader.

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APPENDIX

Proposition A.1

Let $P(\mathbf{i}) = P_1(\mathbf{i}) + P_2(\mathbf{i})$. Then for $P_1(\mathbf{i}),\ P_2(\mathbf{i}),\ P(\mathbf{i}) \neq 0$:

$$\mathbf{D}(P) = a_1\mathbf{D}(P_1) + a_2\mathbf{D}(P_2), \tag{A.1}$$

$$\mathbf{C}(P) = a_1\mathbf{C}(P_1) + a_2\mathbf{C}(P_2), \tag{A.2}$$

where $a_j = P_j(\mathbf{i})/P(\mathbf{i})$.

Proof

Let d_j denote differentiation with respect to i_j. Then:

$$d_jP = d_jP_1 + d_jP_2,$$

$$d_{jk}P = d_{jk}P_1 + d_{jk}P_2.$$

Dividing by $P(\mathbf{i})$ completes the proof. □

Corollary A.1

Let $P(\mathbf{i}) = P_1(\mathbf{i}) + P_2(\mathbf{i})$ and $\mathbf{N} \neq \mathbf{0}$ be given. Then for $P_1(\mathbf{i}),\ P_2(\mathbf{i}),\ P(\mathbf{i}) \neq 0$:

$$\mathbf{D}_N(P) = a_1\mathbf{D}_N(P_1) + a_2\mathbf{D}_N(P_2), \tag{A.3}$$

$$\mathbf{C}_N(P) = a_1\mathbf{C}_N(P_1) + a_2\mathbf{C}_N(P_2), \tag{A.4}$$

where $a_j = P_j(\mathbf{i})/P(\mathbf{i})$.

Proof

Applying (2.7) and (2.8) to Proposition A.1, the result follows. □

Proposition A.2

Let $P(\mathbf{i}) = P_1(\mathbf{i})\,P_2(\mathbf{i})$. Then for $P(\mathbf{i}) \neq 0$:

$$\mathbf{D}(P) = \mathbf{D}(P_1) + \mathbf{D}(P_2), \tag{A.5}$$

$$\mathbf{C}(P) = \mathbf{C}(P_1) + \mathbf{C}(P_2) + \mathbf{D}(P_1)^T\mathbf{D}(P_2) + \mathbf{D}(P_2)^T\mathbf{D}(P_1), \tag{A.6}$$

where $\mathbf{D}^T$ is the column matrix transpose of the row matrix $\mathbf{D}$.

Proof

Let d_j be defined as above, then:

$$d_jP = P_1(d_jP_2) + (d_jP_1)P_2,$$

$$d_{jk}P = (d_{jk}P_1)P_2 + P_1(d_{jk}P_2) + (d_jP_1)(d_kP_2) + (d_jP_2)(d_kP_1).$$

Hence,

$$D_j(P) = D_j(P_1) + D_j(P_2),$$

$$C_{jk}(P) = C_{jk}(P_1) + C_{jk}(P_2) + D_j(P_1)D_k(P_2) + D_j(P_2)d_k(P_1). \quad \square$$

Corollary A.2

Let $P(\mathbf{i}) = P_1(\mathbf{i})P_2(\mathbf{i})$ and $\mathbf{N} \neq \mathbf{0}$ be given. Then for $P(\mathbf{i}) \neq 0$:

$$\mathbf{D}_N(P) = \mathbf{D}_N(P_1) + \mathbf{D}_N(P_2), \tag{A.7}$$

$$\mathbf{C}_N(P) = \mathbf{C}_N(P_1) + \mathbf{C}_N(P_2) + 2\mathbf{D}_N(P_1)\mathbf{D}_N(P_2). \tag{A.8}$$

Proof

Applying (2.7) and (2.8) to Proposition A.2, the result follows. $\square$

Proposition A.3

Let $P(\mathbf{i}) = 1/Q(\mathbf{i})$, $Q(\mathbf{i}) \neq 0$. Then:

$$\mathbf{D}(P) = -\mathbf{D}(Q), \tag{A.9}$$

$$\mathbf{C}(P) = -\mathbf{C}(Q) + 2\mathbf{D}(Q)^T\mathbf{D}(Q). \tag{A.10}$$

Proof

As above,

$$d_jP = -d_jQ/Q^2,$$

from which (A.9) follows. Similarly,

$$d_{jk}P = -d_{jk}Q/Q^2 + 2(d_jQ)\,(d_kQ)/Q^3,$$

from which (A.10) follows. $\square$

Corollary A.3

Let $P(\mathbf{i})=1/Q(\mathbf{i})$, $Q(\mathbf{i})\neq 0$ and $\mathbf{N}\neq\mathbf{0}$ be given. Then:

$$D_N(P) = -D_N(Q), \tag{A.11}$$

$$C_N(P) = -C_N(Q) + 2D_N^2(Q). \tag{A.12}$$

Proof

Immediate. □

Proposition A.4

Let $P(\mathbf{i})=P_1(\mathbf{i})/P_2(\mathbf{i})$, $P_2(\mathbf{i})\neq 0$. Then for $P(\mathbf{i})\neq 0$:

$$\mathbf{D}(P) = \mathbf{D}(P_1) - \mathbf{D}(P_2), \tag{A.13}$$

$$\mathbf{C}(P) = \mathbf{C}(P_1) - \mathbf{C}(P_2) + \mathbf{D}(P_2)^T[\mathbf{D}(P_2) - \mathbf{D}(P_1)] + [\mathbf{D}(P_2) - \mathbf{D}(P_1)]^T\mathbf{D}(P_2). \tag{A.14}$$

Proof

Combining Propositions A.2 and A.3,

$$\mathbf{D}(P) = \mathbf{D}(P_1) + \mathbf{D}(1/P_2) = \mathbf{D}(P_1) - \mathbf{D}(P_2),$$

$$\begin{aligned}\mathbf{C}(P) &= \mathbf{C}(P_1) + \mathbf{C}(1/P_2) + \mathbf{D}(P_1)^T\mathbf{D}(1/P_2) + \mathbf{D}(1/P_2)^T\mathbf{D}(P_1)\\ &= \mathbf{C}(P_1) - \mathbf{C}(P_2) + 2\mathbf{D}(P_2)^T\mathbf{D}(P_2) - \mathbf{D}(P_1)^T\mathbf{D}(P_2)\\ &\quad - \mathbf{D}(P_2)^T\mathbf{D}(P_1).\end{aligned}$$ □

Corollary A.4

Let $P(\mathbf{i})=P_1(\mathbf{i})/P_2(\mathbf{i})$, $P_2(\mathbf{i})\neq 0$ and $\mathbf{N}\neq\mathbf{0}$ be given. Then for $P(\mathbf{i})\neq 0$:

$$D_N(P) = D_N(P_1) - D_N(P_2), \tag{A.15}$$

$$C_N(P) = C_N(P_1) - C_N(P_2) + 2D_N(P_2)[D_N(P_2) - D_N(P_1)]. \tag{A.16}$$

Proof

Immediate. □

Proposition A.5

Let $\mathbf{A}:\mathbf{i}\to\mathbf{j}$ be a smooth transformation from $\mathbf{R}^m$ to $\mathbf{R}^n$. Let $Q(\mathbf{j})$ be a price function and define $P(\mathbf{i})=Q(\mathbf{Ai})$. Then:

$$\mathbf{D}(P;\mathbf{i}) = \mathbf{D}(Q;\mathbf{Ai})\mathbf{J}[\mathbf{A}(\mathbf{i})], \tag{A.17}$$

$$\mathbf{C}(P;\mathbf{i}) = \mathbf{J}[\mathbf{A}(\mathbf{i})]^T\mathbf{C}(Q;\mathbf{Ai})\mathbf{J}[\mathbf{A}(\mathbf{i})] - \mathbf{D}(Q;\mathbf{Ai})\cdot\mathbf{H}[\mathbf{A}(\mathbf{i})], \tag{A.18}$$

where $\mathbf{J}[\mathbf{A}(\mathbf{i})]_{jk}=\partial A_j/\partial i_k$ is the $n \times m$ Jacobian matrix of $\mathbf{A}$, and $\mathbf{H}[\mathbf{A}(\mathbf{i})]_{jkl} = \partial^2 A_j/\partial i_k \partial i_l$ is the $n \times m \times m$ Hessian 'matrix' of $\mathbf{A}$.

Proof

Applying the chain rule:

$$d_k P(\mathbf{i}) = \sum_j d_j Q(\mathbf{Ai})\, d_k A_j(\mathbf{i}) = \mathbf{dQ} \cdot d_k \mathbf{A},$$

from which (A.17) follows. Taking second derivatives:

$$d_{lk}P(\mathbf{i}) = \sum_j \sum_i d_{ij}Q(\mathbf{Ai})\, d_l A_i(\mathbf{i})\, d_k A_j(\mathbf{i}) + \sum_j d_j Q\,(\mathbf{Ai})\, d_{lk} A_j(\mathbf{i})$$

$$= (d_l\mathbf{A})^T[\mathbf{d^2Q}]\, d_k\mathbf{A} + \mathbf{dQ} \cdot d_{lk}\mathbf{A},$$

from which we obtain (A.18). □

Corollary A.5

Let $\mathbf{A}$, $Q(\mathbf{j})$ and $P(\mathbf{i})$ be as in Proposition A.5, and $\mathbf{N}\neq\mathbf{0}$ be a given direction vector in $\mathbf{R}^m$. Also, let $\mathbf{M}$ and $\mathbf{M}'$ be defined in $\mathbf{R}^n$ by:

$$\mathbf{M} = \mathbf{J}[\mathbf{A}(\mathbf{i})]\mathbf{N}, \tag{A.19}$$

$$\mathbf{M}' = \mathbf{N}^T\mathbf{H}[\mathbf{A}(\mathbf{i})]\mathbf{N}. \tag{A.20}$$

Then:

$$D_N(P) = D_M(Q), \tag{A.21}$$

$$C_N(P) = C_M(Q) - D_{M'}(Q). \tag{A.22}$$

Proof

Using (2.7), (A.21) follows immediately from (A.17). Similarly, (2.8) makes the first term on the right of (A.22) clear from (A.18). For the second term, we have:

$$\begin{aligned}
-\mathbf{N}^T\mathbf{D}(Q;\mathbf{Ai})\mathbf{H}[\mathbf{A}(\mathbf{i})]\mathbf{N} &= \sum_j d_j Q(\mathbf{Ai}) \sum_{lk} d_{lk}\mathbf{A}_j(\mathbf{i})\, n_l n_k / Q(\mathbf{Ai}) \\
&= \mathbf{dQ} \cdot [\mathbf{N}^T\mathbf{H}[\mathbf{A}(\mathbf{i})]\mathbf{N}]/Q(\mathbf{Ai}) \\
&= -\mathbf{D}(Q) \cdot \mathbf{M}' \\
&= -\,D_{M'}(Q). \ \Box
\end{aligned}$$

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The work of science is to substitute facts for appearances and demonstrations for impressions.—RUSKIN

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